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Plane elementary bipartite graphs ☆

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Abstract

A connected graph is called elementary if the union of all perfect matchings forms a connected subgraph. In this paper we mainly study various properties of plane elementary bipartite graphs so that many important results previously obtained for hexagonal systems are treated in a unified way. Firstly, we show that a plane bipartite graph G is elementary if and only if the boundary of each face (including the infinite face) is an alternating cycle with respect to some perfect matching of G. For a plane bipartite graph G all interior vertices of which are of the same degree, a stronger result is obtained; namely, G is elementary if and only if the boundary of the infinite face of G is an alternating cycle with respect to some perfect matching of G. Second, the concept of the Z-transformation graph $Z(G)$ of a hexagonal system G (whose vertices represent the perfect matchings of G) is extended to a plane bipartite graph G and some results analogous to those for hexagonal systems are obtained. A peripheral face f of G is called reducible if the removal of the internal vertices and edges of the path that is the intersection of f and the exterior face of G results in a plane elementary bipartite graph. Thirdly, we obtain the reducible face decomposition for plane elementary bipartite graphs. Furthermore, sharp upper and lower bounds for the number of reducible faces are derived. Conversely, we can construct any plane elementary bipartite graphs by adding new peripheral faces one by one. As applications of this approach, we give simple construction methods for several types of plane elementary bipartite graphs G that contain a forcing edge (which belongs to exactly one perfect matching of G) and whose Z-transformation graphs $Z(G)$ contain vertices of degree one.

Keywords: Plane bipartite graph; Elementary bipartite graph; Perfect matching; Z-transformation graph; Ear decomposition; Forcing edge

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1. Introduction

We take the basic terminology from [18]. Let G be a connected bipartite graph with p vertices and q edges. A *perfect matching* (or 1-*factor*) of G is a set of independent edges of G covering all vertices of G. An edge of G is termed *allowed* if it lies in some perfect matching of G and *forbidden* otherwise. A graph G is said to be *elementary* if all its allowed edges form a connected subgraph of G. The investigation of elementary bipartite graphs has a long history. As early as in 1915, König [16] had employed this concept in studying the decomposition of a determinant. After nearly half a century, Hetyei [11] formally used the term "elementary" for this concept and obtained various properties of elementary bipartite graphs. It is well known that a connected bipartite graph G is elementary if and only if G is 1-extendable, i.e., each edge of G belongs to a perfect matching of G. A graph G is said to be *n-extendable* ($n \leqslant (p-2)/2$) if any matching consisting of n edges is contained in a perfect matching [20]. Plummer showed that an n-extendable graph is $(n+1)$-connected (thus an elementary bipartite graph is 2-connected), and $(n-1)$-extendable [19]; a planar graph is not 3-extendable [20]. Lovász and Plummer [17] gave some further results by studying minimal elementary bipartite graphs. For more details on this field see the excellent book "Matching Theory" due to Lovász and Plummer [18] and a survey [21] with the references cited therein.

On the other hand, for special types of plane bipartite graphs (for instance, hexagonal systems and polyomino graphs) some problems involving perfect matchings that are of physico-chemical relevance have been studied extensively and intensively [3,6,7,15,38]. A face of a 2-connected plane bipartite graph G is called *resonant* if its boundary is an alternating cycle with respect to some perfect matching of G. A hexagonal system (or polyhex graph) is a special connected plane bipartite graph without cut vertices each interior face of which is surrounded by a regular hexagon of side length one [24]. Some chemists are much interested in characterizing those hexagonal systems that are normal, i.e. such that each of their unit hexagons is resonant. It turns out that a hexagonal system is normal if and only if it is elementary, or equivalently, if and only if its exterior face is resonant [12,13,14,26]. Such concepts originally defined for hexagonal systems have been extended to generalized hexagonal systems (which may contain the so-called "holes", interior faces larger than a unit hexagon) [31]. For studies of a further problem, the so-called "k-resonant problem", see [5,36,37]. This concept of "resonance" is related to the concept of an aromatic sextet [2]. In connection to resonant hexagons (or aromatic sextets), Z-transformation graphs of perfect matchings of hexagonal systems H were defined [27,28]. It was shown that the degree sum of the Z-transformation graph of a hexagonal system can be used to estimate the resonance energies of the corresponding benzenoid hydrocarbons [30]. The practicability of a hexagon-addition construction for normal hexagonal systems was once conjectured [3] and proved by He and He [10] and Zheng [38].

Because of the variety of molecular structures, we need to consider general plane bipartite graphs, which correspond to conjugated alternate compounds. Some applications of such graphs to chemistry have been made [4,22,24]. In this paper we study plane elementary bipartite graphs so that some important results previously obtained for hexagonal systems, polyomino graphs and so on, are treated in a unified way. Throughout this paper all the plane bipartite graphs considered have *more than two vertices* and the vertices are colored properly black and white such that the two end vertices of each edge receive different colors. Let G be a plane bipartite graph with a perfect matching M. The sets of vertices and edges of G are denoted by $V(G)$ and $E(G)$, respectively. The boundary of a finite face of G is called a *ring* if it is a cycle of G. A cycle C (resp. a path P) of G is called *M-alternating* if the edges of C (resp. P) appear alternately in M and $E(G)\backslash M$. A face f of G is said to be *resonant* if G has a perfect matching M such that the boundary of f is an M-alternating cycle. The symmetric difference of two finite sets M_1 and M_2 is defined by $M_1 \oplus M_2 := (M_1 \cup M_2)\backslash(M_1 \cap M_2)$. This binary operation is associative and commutative.

First, we show that a plane bipartite graph G is elementary if and only if each face (including the infinite face) is resonant. For a plane bipartite graph G all interior vertices of which are of the same degree, a stronger result is obtained; namely, G is elementary if and only if the infinite face of G is resonant. Secondly, the concept of the Z-transformation graph of a hexagonal system is extended to a plane bipartite graph, and for plane (weakly) elementary bipartite graphs some results are obtained similar to those for hexagonal systems. By the *periphery* of G we mean the boundary of the exterior face of G. By a face of G we always mean a finite face of G unless specified otherwise. A face f of G is called a *peripheral face* of G if the peripheries of f and G have a path of positive length in common. A peripheral face f of G is called a *reducible face* of G if the removal of the internal vertices and edges of the path that is the intersection of f and the exterior face of G results in a plane elementary bipartite graph. In Section 4 we obtain the reducible face decomposition for plane elementary bipartite graphs. Furthermore, sharp upper and lower bounds for the number of reducible faces are derived. Conversely, we can construct any plane elementary bipartite graph by adding new peripheral faces one by one. By virtue of the above approach, in Section 5 we can easily construct all plane elementary bipartite graph whose Z-transformation graphs have vertices of degree one, especially are paths. An edge of a plane bipartite graph is called a *forcing edge* if it lies in precisely one perfect matching. Eventually, we give a characterization for plane bipartite graphs having a forcing edge. A very simple construction is presented.

2. Resonance of plane graphs

A subgraph H of G is said to be *nice* if $G - V(H)$ has a perfect matching. Let G be a bipartite graph with a perfect matching M and a cycle C. If the edges of C appear alternately in M and $E(G)\backslash M$, then we say that C is an *M-alternating cycle*. It is obvious that a cycle of G is nice if and only if there is a perfect matching M of G such that C is an M-alternating cycle. If the edges of a cycle C appear alternately in two matchings M_1 and M_2, we say that C is an *(M_1, M_2)-alternating cycle*. By

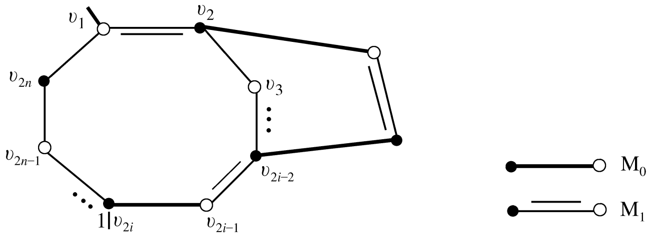


Fig. 1.

the same fashion, we may define M- and (M_1,M_2)-alternating paths. If M_1 and M_2 are two different perfect matchings, then the symmetric difference $M_1 \oplus M_2$ consists of mutually disjoint (M_1,M_2)-alternating cycles. Let G be a plane bipartite graph and C the boundary of a face f of G. If G has a perfect matching M such that C is an M-alternating cycle, then C and f will be called an *M-resonant cycle* and *face*, respectively. In other words, a face is resonant if and only if the boundary of it is a nice cycle.

Lemma 2.1. *Let M be a perfect matching of a graph G and C an M-alternating cycle of G. Then $M \oplus C(=M \oplus E(C))$ is also a perfect matching of G and C is an $(M \oplus C)$-alternating cycle of G. Thus each edge of any M-alternating cycle is allowed.*

Proof. The proof is obvious. $\square$

Lemma 2.2. *Let G be a connected plane bipartite graph with perfect matchings. Assume that the cycle C of G lies in the boundary of some face of G. If $\frac{1}{2}|V(C)|$ independent edges of C are allowed, then C is a nice cycle.*

Proof. Without loss of generality, assume that C lies in the boundary of a finite face (interior face). We label clockwise the vertices of C by $v_1, v_2, \ldots, v_{2n}$, where $2n = |V(C)|$ (see Fig. 1). We make the convention that $v_1, v_3, \ldots, v_{2n-1}$ are colored white and $v_2, v_4, \ldots, v_{2n}$ black. Let

$$I = \{v_1v_2, v_3v_4, \ldots, v_{2n-1}v_{2n}\}$$

consist of n independent edges of C each of which is allowed. Among all perfect matchings M of G, we choose one containing as many edges of I as possible, M_0, say.

We claim that $I \subseteq M_0$.

Suppose that $I \nsubseteq M_0$. Set $I_1 = I \cap M_0$. Without loss of generality assume that $v_1v_2 \notin I_1$. Since v_1v_2 is an allowed edge of G, G has a perfect matching M_1 such that $v_1v_2 \in M_1$. $M_1 \oplus M_0$ has an (M_1,M_0)-alternating cycle (say C_1) containing v_1v_2. In fact,

$$E(C_1) \cap I_1 = \emptyset.$$

Otherwise, assume that $v_{2i-1}v_{2i} \in E(C_1) \cap I_1$ for some i. We now traverse the cycle C_1 starting with v_1 and then entering into v_2 through edge v_1v_2. From the planarity of G, from the assumption that C lies in the boundary of a face and from the fact that an (M_0, M_1)-alternating cycle cannot intersect itself we immediately deduce that we first enter v_{2i-1} then reach v_{2i} through edge $v_{2i-1}v_{2i}$. An (M_0, M_1)-alternating path P from v_2 to v_{2i-1} must have the property that two end edges of P belong to M_0 and M_1 separately. Hence P is a path of even length, and two end vertices should be of the same color, which contradicts that v_2 and v_{2i-1} are of different colors.

Since C_1 is an (M_0, M_1)-alternating cycle, $M_0 \oplus C_1$ is another perfect matching of G by Lemma 1. Moreover, since $E(C_1) \cap I_1 = \emptyset$ and $v_1v_2 \notin M_0$, then $I_1 \cup \{v_1v_2\} \subseteq M_0 \oplus C_1$. Hence $|(M_0 \oplus C_1) \cap I| \geqslant |M_0 \cap I| + 1$, which contradicts the choice for M_0. So our assertion is verified. Namely, C is an M_0-alternating cycle and thus a nice cycle. □

As an immediate consequence, we have

Corollary 2.3. *Let G be a plane bipartite graph with perfect matchings. Assume that a cycle $C = v_1v_2 \cdots v_{2n}v_1$ of G lies in the boundary of a face of G. If an edge $v_{2i-1}v_{2i}$ is a forbidden edge of G, then there exists j such that $v_{2j}v_{2j+1}$ is also a forbidden edge of G.*

Theorem 2.4. *Let G be a plane bipartite graph with more than two vertices. Then each face of G is resonant if and only if G is elementary.*

Proof. Assume that G is an elementary bipartite graph. Then G is 2-connected and each edge is allowed [18]. By Lemma 2.2 we know that the boundary of each face of G is a nice cycle and each face is resonant. Conversely, assume that each face of G is resonant. By the definition, the boundary of each face (including the infinite face) of G is a nice cycle. It is easily seen that G is connected. Moreover, each edge of G lies on the boundary of a face and is thus allowed by Lemma 1. Hence G is elementary. □

To discuss an alternative criterion for determining whether a given plane bipartite graph is elementary or not, we need the concept of the (geometric) dual of a plane graph. Let G be a plane bipartite graph. Denote by G^* the dual graph of G (for the definition of dual graph, the reader is referred to [1]). Let C^* be a cycle of G^* (C^* may be a loop or consists of a pair of parallel edges). Let $\mathscr{C}$ denote the set of edges of G intersecting an edge of C^*. If G is connected, it is easy to see that $G - \mathscr{C}$ contains exactly two components.

Definition 2.5. Let G be a plane bipartite graph. A cycle C^* of G^* is called an *elementary closed cut line* and $\mathscr{C}$ *an elementary edge cut* of G, if all edges of $\mathscr{C}$ are incident with white vertices of one of the components of $G - \mathscr{C}$ (denoted by $G_w(\mathscr{C})$ and called the white bank of $\mathscr{C}$) and black vertices of the other one (denoted by $G_b(\mathscr{C})$ and called the black bank of $\mathscr{C}$).

Let H be a subgraph of G and let $w(H)$ and $b(H)$ denote the numbers of white and black vertices of H, respectively. The following result is obvious.

Lemma 2.6. *Let a connected plane bipartite graph G have a perfect matching and let $\mathscr{C}$ be an elementary edge cut. Then all edges of $\mathscr{C}$ are forbidden if and only if* $b(G_w(\mathscr{C})) = w(G_w(\mathscr{C}))$. □

Corollary 2.3 and Lemma 2.6 can be used to produce the following result.

Theorem 2.7. *Let a connected plane bipartite graph G have a perfect matching. Then G is not elementary if and only if G has an elementary edge cut $\mathscr{C}$ such that $b(G_w(\mathscr{C})) = w(G_w(\mathscr{C}))$, i.e., all edges of $\mathscr{C}$ are forbidden.*

Proof. Omitted. □

For a special type of plane bipartite graphs, some stronger results will be described.

Definition 2.8. Let a connected plane bipartite graph G have a perfect matching. G is said to be *weakly elementary* if for each nice cycle C of G the interior of C has at least one allowed edge of G that is incident with a vertex of C whenever the interior of C contains an edge of G.

Clearly, every plane elementary bipartite graph is weakly elementary.

Theorem 2.9. *Let a connected plane bipartite graph G have a perfect matching. Then G is weakly elementary if and only if for each nice cycle C the subgraph of G consisting of C together with its interior is elementary.*

Proof. It suffices to prove the necessity. Suppose that a connected plane bipartite graph G is weakly elementary and let C be a nice cycle of G. Denote by $I[C]$ the subgraph of G consisting of C together with the interior. Obviously, $I[C]$ is connected. We now show that $I[C]$ is elementary by induction on the number m of edges contained in the interior of C. If $m = 0$, this is trivially true. So assume that $m \geqslant 1$. Then the interior of C has an allowed edge e incident with a vertex u of C. Let M and M' be perfect matchings of G such that C is an M -alternating cycle and $e \in M'$. Then $M \oplus M'$ has a cycle containing e. Let $P = ue \cdots v$ be a path of the cycle such that only end vertices lie on C (see Fig. 2). C and P form two cycles C_1 and C_2 such that $C_1 \cap C_2 = P$. Without loss of generality, suppose that C_1 is an M-alternating cycle and C_2 is thus an $M \oplus C_1$-alternating cycle (note that C_1 and C_2 are nice cycles). By the induction hypothesis we know that $I[C_1]$ and $I[C_2]$ are elementary. Hence $I[C]$ is elementary. □

Theorem 2.10. *Assume that a 2-connected plane bipartite graph G is weakly elementary. Then the following statements are equivalent*:

(i) *G is elementary,*
(ii) *each interior face of G is resonant,*
(iii) *the exterior face of G is resonant.*

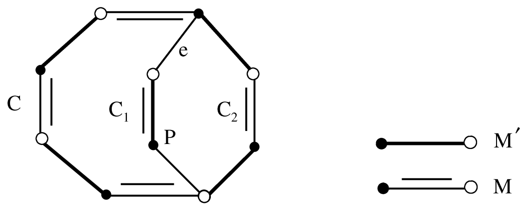


Fig. 2.

Proof. (i) $\Leftrightarrow$ (ii): This is obvious by Theorem 2.4.
(i) $\Leftrightarrow$ (iii): This follows from Theorems 2.4 and 2.9. $\square$

Theorem 2.11. *Let G be a connected plane bipartite graph that has a perfect matching. If the interior vertices of G (not lying on the boundary of the infinite face of G) all have the same degree then G is weakly elementary.*

Proof. Let C be any nice cycle of G. Denote by $H=I[C]$ the subgraph of G consisting of C together with its interior. Obviously, H is connected. Suppose that H is not elementary. By Theorem 2.7 H must have an elementary edge cut $\mathscr{C}$ such that all edges of $\mathscr{C}$ are forbidden. Obviously, $\mathscr{C}\cap E(C)=\emptyset$. Then all the vertices of the black or white bank of $\mathscr{C}$ (say black bank $H_b(\mathscr{C})$) must be interior vertices of H and thus of G. It is obvious that

$$d\,w(H_b(\mathscr{C}))+|\mathscr{C}|=d\,b(H_b(\mathscr{C})),$$

where $d>1$ is the degree of the interior vertices of G. Since $|\mathscr{C}|\geqslant 1$ implies that $b(H_b(\mathscr{C}))>w(H_b(\mathscr{C}))$, $\mathscr{C}$ has an allowed edge of H, a contradiction. $\square$

Many plane bipartite graphs with important applications, such as hexagonal systems, polyomino graphs, etc., have the property that their interior vertices all have the same degree and, therefore, are weakly elementary. Thus, certain results previously obtained for hexagonal systems [12,26] and polyomino graphs [32,34] only have been described in Theorem 2.10 in a unifying manner or more general way.

3. Z-transformation graphs

The concept of Z-transformation graph of a hexagonal system was first introduced in Refs. [27,28]. We now extend this concept to plane bipartite graphs.

Definition 3.1. Let G be a plane bipartite graph. The *Z-transformation graph* of G, denoted by $Z(G)$, is defined as the graph whose vertices represent the perfect matchings of G where two vertices are adjacent if and only if the symmetric difference of the corresponding two perfect matchings consists exactly of the boundary of some finite face of G.

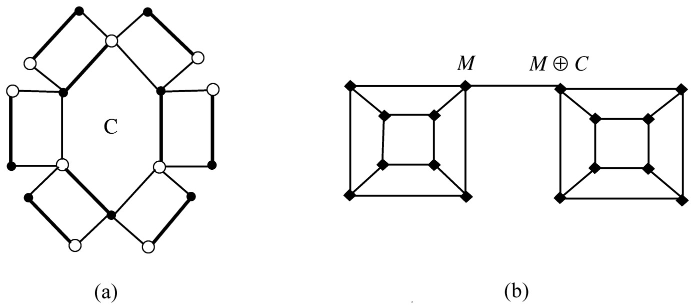


Fig. 3. (a). A plane elementary bipartite graph G with a perfect matching M; (b). The Z-transformation graph $Z(G)$ with two cut vertices M and $M \oplus C$.

Let G be the graph of Fig. 3. It is easy to see that $Z(G)$ consists of two disjoint cubes and an edge connecting them. This example shows that the connectivity of the Z-transformation graph of a plane elementary bipartite graph is not necessarily equal to its minimum degree. For a plane bipartite graph G with a forbidden edge, $Z(G)$ is not necessarily connected. For an example, see [32]. In the following, we derive some basic properties of Z-transformation graphs.

Theorem 3.2. *Let G be a plane bipartite graph with a perfect matching. Then $Z(G)$ is a bipartite graph.*

Proof. Assume that $Z(G)$ has a cycle $M_1M_2\cdots M_kM_1$, where $M_{i+1}=M_i \oplus s_i$ and s_i is a ring of G and also an M_i-alternating cycle, $i=1,2,\ldots,k$ (the subscripts modulo k).

Let h be the boundary of a face (the infinite face is allowed) of G. Denote by $\delta(h)$ the number of times h appears in the sequence $s_1s_2\cdots s_k$. Let h_1 and h_2 denote the boundaries of two faces of G having an edge e in common. We assert that $\delta(h_1) + \delta(h_2) \equiv 0 \pmod 2$. The cycle $M_1M_2\cdots M_kM_1$ of $Z(G)$ implies that $M_1 = M_1 \oplus s_1 \oplus s_2 \oplus \cdots \oplus s_k$; that is, $s_1 \oplus s_2 \oplus \cdots \oplus s_k = \emptyset$. Hence there are exactly an even number (zero is also allowed) of terms in the sequence $s_1,s_2,\ldots,s_k$ that contain the edge e. Since h_1 and h_2 are the only face boundaries of G containing the edge e, the assertion is verified. Moreover, for a face boundary h of G that is not a ring of G, such as the boundary of the exterior face of G, it is obvious that $\delta(h)=0$. By the connectedness of the dual graph of G it is easily shown that all $\delta(h)$ have the same parity, and thus $\delta(h) \equiv 0 \pmod 2$ for all rings h of G. Therefore,

$$k = \sum_h \delta(h) \equiv 0 \pmod 2.$$

So every cycle of $Z(G)$ is of even length, and thus $Z(G)$ is bipartite. $\square$

Theorem 3.3. *Let G be a plane elementary bipartite graph. Then $Z(G)$ is connected.*

Proof. Let M_1 and M_2 be any perfect matchings of G. We prove that $Z(G)$ has a path connecting M_1 and M_2 by induction on the number m of cycles of $M_1 \oplus M_2$. When

$m=1$, it can be shown that $Z(G)$ has a path connecting M_1 and M_2 by arguments similar to those used in the proof of Theorem 2.9 (using induction on the number of edges in the interior of the cycle). If $m>1$, let C be a cycle of $M_1 \oplus M_2$. By the induction hypothesis, $Z(G)$ has a path connecting M_1 and $M_2 \oplus C$ and a path connecting $M_2 \oplus C$ and M_2. The theorem follows. □

Corollary 3.4. *Let G be a plane elementary bipartite graph with a perfect matching M and let C be an M-alternating cycle. Then there exists an M-resonant face in the interior of C.*

Proof. Let $I[C]$ denote a subgraph of G consisting of C together with the interior of C. By Theorem 2.9 $I[C]$ is a nice subgraph of G and elementary. The restriction of M on $I[C]$ is also a perfect matching of $I[C]$ and denoted by M_C. By Theorem 3.3, $Z(I[C])$ is connected and thus the degree of M_C is not less than one. Hence G has an M-resonant face in the interior of C. □

By Corollary 3.4 we can prove the following result.

Theorem 3.5. *Let G be a plane elementary bipartite graph. Then $Z(G)$ has at most two vertices of degree one.*

Moreover, to study the relation between cuts and blocks of $Z(G)$ we consider its block-graph (defined in [25,35]). By orientating each edge of $Z(G)$ the following theorem is obtained.

Theorem 3.6 (Zhang and Zhang [35]). *Let G be a plane elementary bipartite graph. Then the block-graph of $Z(G)$ is a path.*

It is obvious that Theorem 3.5 may be viewed as an immediate consequence of Theorem 3.6. In addition, for two special types of plane bipartite graphs the connectivity of their Z-transformation graphs have been treated. It was shown that the connectivity of the Z-transformation graph of a hexagonal system is equal to its minimum degree [28], and a similar result holds for Z-transformation graphs of polyomino graphs with only two exceptions [32].

Lemma 3.7. *Let G be a plane elementary bipartite graph and M a perfect matching of G. If there exist three distinct M-resonant finite faces, then there are two of them whose boundaries are disjoint.*

Proof. Assume that s_1, s_2 and s_3 are the boundaries of mutually distinct M-resonant finite faces, which are all M-alternating cycles. We shall prove that at least two of s_1, s_2 and s_3 are disjoint. Without loss of generality, we may assume that both s_1 and s_2 intersect s_3, and all M-matched edges (i.e. those edges that belong to M) of s_3 go from white vertices to black vertices following the clockwise orientation of s_3. Let e

be an M-matched edge of G belonging to both s_1 and s_3. Since s_1 lies in the exterior of s_3, edge e and thus all M-matched edges of s_1 go from their black end vertices to their white end vertices in the clockwise orientation of s_1. Similarly, we see that all M-matched edges of s_2 go from black end vertices to white end vertices in the clockwise orientation of s_2. Similar to the above arguments, these results imply that s_2 and s_1 are disjoint. □

Theorem 3.8. *Let G be a plane elementary bipartite graph. If $Z(G)$ has a vertex of degree no less than 3 then the girth (minimum cycle length) of $Z(G)$ is 4; otherwise $Z(G)$ is a path.*

Proof. (1) Assume that M_1 is a perfect matching of G whose degree in $Z(G)$ is at least 3. Let s_1, s_2 and s_3 be the boundaries of three mutually distinct M_1-resonant finite faces, which are all M_1-alternating cycles. By Lemma 3.7 two of s_1, s_2 and s_3 (say s_1 and s_2) must be disjoint. We can construct the following cycle of length 4: $M_1M_2M_3M_4M_1$, where $M_2 = M_1 \oplus s_1, M_3 = M_2 \oplus s_2, M_4 = M_3 \oplus s_1$ and $M_1 = M_4 \oplus s_2$, which is a cycle of the minimum length since $Z(G)$ is bipartite. Thus if $Z(G)$ has a vertex of degree $\geqslant 3$ then it is of girth 4.

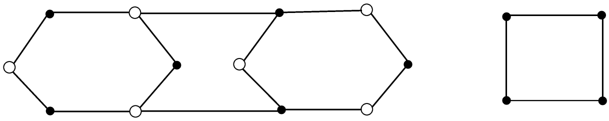
(2) Assume that the degree of each vertex of $Z(G)$ is not more than 2. We shall prove that $Z(G)$ is a path. Otherwise, by the connectedness $Z(G)$ consists of exactly one cycle, say $M_1M_2 \cdots M_kM_1$, where $M_{i+1} = M_i \oplus s_i$ and s_i is the boundary of some M_i-resonant finite face (subscripts modulo k). Because G is a plane elementary bipartite graph, by Theorem 2.4 each face of G is resonant. Hence the boundary of each finite face, i.e. each ring, of G is contained in the sequence $s_1s_2 \cdots s_k$.

Claim 1. *For such a finite circular sequence $s_1s_2 \cdots s_ks_1$, any two consecutive s_i and s_{i+1} are different and intersecting.*

Proof. For any given i ($1 \leqslant i \leqslant k$), it is obvious that $s_i \neq s_{i+1}$. Suppose that s_i and s_{i+1} are disjoint. As shown in part (1) of this proof, $Z(G)$ contains and is thus a cycle of length 4. So $k = 4$ and the sequence $s_1s_2s_3s_4$ contains only two different rings s_i and s_{i+1}. But G contains another ring that is neither s_i nor s_{i+1}, which contradicts that each ring of G is contained in the sequence $s_1 \cdots s_k$. □

Claim 2. *Let $s_1s_2 \cdots s_r$ ($k \geqslant r \geqslant 3$) be a segment of the sequence $s_1s_2 \cdots s_k$. If $s_1 = s_r \neq s_i$ for any $1 < i < r$ and there is a path of s_1 with positive length the internal of which is disjoint with all rings s_i ($2 \leqslant i \leqslant r-1$), then $s_1s_2 \cdots s_r$ has also a segment $s_2 \cdots s_m$ ($m < r$) with length $\geqslant 3$ such that $s_2 = s_m \neq s_i$ for all $2 < i < m$.*

Proof. By assumption we have that $s_1 = s_r = M_1 \oplus M_2 = M_r \oplus M_{r+1}$, and thus s_1 is also an M_r-alternating cycle. Let P be a path of s_1 with positive length such that the internal vertices and the s_i are disjoint for all $1 < i < r$. Obviously, in the process of transformations from M_2 to M_r, the way that the edges of P are matched remains



(a) (b)

Fig. 4. (a) A plane weakly elementary bipartite graph and (b) its Z-transformation graph.

unchanged. Since s_1 is both M_2- and M_r-alternating, $E(s_1)\cap M_2 = E(s_1)\cap M_r$. Furthermore, we can prove that there exists a j, $2<j<r$, such that $s_j = s_2$; otherwise $E(s_1\cap s_2)\cap E(s_i)=\emptyset$ for all $2<i<r$. Then the way that the edges of $s_1\cap s_2$ are matched also remains unchanged when M_3 is transformed into M_r. Hence $s_1\cap s_2$ is (M_2, M_r)-alternating, which contradicts that $E(s_1)\cap M_2 = E(s_1)\cap M_r$. We now choose the minimum subscript m such that $s_2 = s_m$ $(m>2)$, as claimed. □

Since the boundary of each finite face must appear in $s_1 s_2\cdots s_k$, without loss of generality, assume that s_1 is the boundary of a peripheral face of G. Since $Z(G)$ is a cycle, s_1 appears an even number of times (at least twice) in this sequence, thus we can find a segment $s_1 s_2\cdots s_r$ $(s_1 = s_r)$ satisfying the condition of Claim 2. By Claim 2, there exists a segment $s_2\cdots s_{r'}$ $(r'<r)$ with length $\geqslant 3$ and $s_2 = s_{r'} \neq s_i$ for all $3<i<r'$. It is obvious that each component of $s_1\cap s_2$ is a path of odd length; the internal vertices are disjoint with the rings s_i $(3\leqslant i<r')$. Hence, this segment also satisfies the condition of Claim 2. We continue this procedure and eventually show that the sequence $s_1 s_2\cdots s_k$ is infinite, which contradicts the finiteness of $Z(G)$. This contradiction shows that $Z(G)$ is a path. □

Let G be a plane weakly elementary bipartite graph. Deleting all forbidden edges of G (if such edges exist), we obtain a subgraph of G consisting of a number of components $G_1, G_2, \dots, G_k$ $(k\geqslant 1)$, which are called the *elementary components* of G, with more than one edge and some K_2's. Each G_i can be regarded as a plane elementary bipartite graph. By Theorems 2.9 and 2.4 a finite face of G is resonant if and only if it is a resonant finite face of some G_i. Thus $Z(G)=Z(G_1)\times Z(G_2)\times\cdots\times Z(G_k)$, where "$\times$" denotes the Cartesian product of graphs. Thus for a plane weakly elementary bipartite graph the above results (Theorems 3.3–3.7) are still valid; Theorem 3.8 implies that $Z(G)$ is not a cycle for a plane elementary bipartite graph G. However, when G is weakly elementary, Theorem 3.8 also holds except for a single case: $Z(G)$ is a cycle of length 4 whenever G has exactly two elementary components that are cycles (for an example, see Fig. 4).

4. Reducible face decomposition

An important property of elementary bipartite graph is the so-called "ear decomposition", the details of which are described below. Let x be an edge. Join its end vertices by a path P_1 of odd length (the so-called “first ear”). We proceed inductively to build a sequence of bipartite graphs as follows: if $G_{r-1}=x+P_1+P_2+\cdots+P_{r-1}$ has already been constructed, add the rth *ear* P_r (of odd length) by joining any two vertices in different colors of G_{r-1} such that P_r have no internal vertices in common with the vertices of G_{r-1}. The decomposition $G_r = x + P_1 + P_2 + \cdots + P_r$ will be called an (*bipartite*) *ear decomposition* of G_r.

Theorem 4.1 (Lovász and Plummer [17,18]). *A bipartite graph is elementary if and only if it has an* (*bipartite*) *ear decomposition.*

An ear decomposition $G=x+P_1+P_2+\cdots+P_r$ of an elementary bipartite graph G can be represented conveniently by the sequence of elementary bipartite graphs $(G_0, G_1, \ldots, G_r(=G))$, where $G_0 = x$ and for $0 \leqslant i \leqslant r$, $G_i = x + P_1 + \cdots + P_i$. It is easy to see that the number r of ears is equal to the cyclomatic number of G, i.e. $r = q - p + 1$. Given any nice subgraph H (defined in Section 2), in particular $H = K_2$, of an elementary bipartite graph G, G has an ear decomposition $G = H + P_1 + \cdots + P_t$ starting with H just as in Theorem 4.1, where for all $1 \leqslant i \leqslant t$, P_i is an ear (i.e., a path of odd length) joining different color classes of $H + P_1 + \cdots + P_{i-1}$.

Definition 4.2. An ear decomposition $(G_1, G_2, \ldots, G_r(=G))$ (equivalently, $G = x + P_1 + P_2 + \cdots + P_r$) of a plane elementary bipartite graph G is called a *reducible face decomposition* (hereafter abbreviated “RFD”) if G_1 is the boundary of an interior face of G and the ith ear P_i lies in the exterior of G_{i-1} such that P_i and a part of the periphery of G_{i-1} surround an interior face of G for all $2 \leqslant i \leqslant r$.

Theorem 4.3. *Let G be a plane elementary bipartite graph other than K_2. Then G has a reducible face decomposition* (*RFD*) *starting with the boundary of any interior face of G.*

Proof. Let G be a plane elementary bipartite graph other than K_2. Let G_1 be the boundary of an arbitrary finite face of G. By Theorem 2.4, each face of G is resonant. So G_1 is a nice subgraph (cycle) of G. If $r = 1$, there is nothing to prove. So we assume that $r \geqslant 2$. For any ear decomposition $(G_1, G_2, \ldots, G_r(=G))$ of G starting with G_1, the second ear P_2 lies in the exterior of G_1. P_2 and a path P of the periphery of G_1 form a cycle, denoted by $P + P_2$, the interior of which lies in the exterior of G_1. The common end vertices of P and P_2 are represented by u and v. Without loss of generality, suppose that the second ear P_2 is minimal among all ear decompositions of G that leave G_1 unchanged, i.e. there is no other second ear in the interior of $P + P_2$. We assert that $P + P_2$ is just the boundary of a finite face of G lying in the exterior of G_1. Otherwise, there exists an edge e in the interior of $P + P_2$ incident with a vertex u' of $P + P_2$. Since G is elementary, it has a perfect matching M_e containing e. Obviously, $G - V(G_1)$ has a perfect matching M such that P_2 is an M-alternating path. The symmetric difference $(M \backslash E(P_2)) \oplus M_e$ has an (M_e, M)-alternating path P'

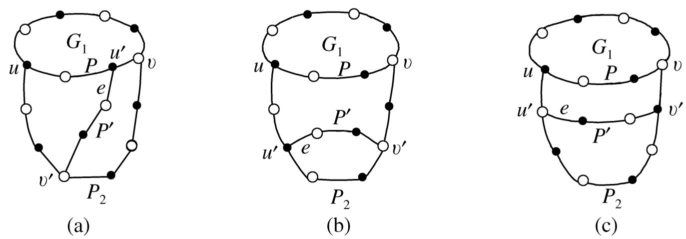


Fig. 5.

containing e in the interior of $P + P_2$, the first and last edges of which belong to M_e. And only the end vertices u' and v' of P' belong to $P + P_2$. We distinguish the following cases.

Case 1: Both u' and v' lie on P. Then we can take P' as the second ear of some ear decomposition of G (since $G_1 + P'$ is a nice subgraph of G) lying in the interior of $P + P_2$, which contradicts the minimality of P_2.

Case 2: Exactly one of u' and v' (say u') belongs to P. So v' belongs to P_2. u' and one of u and v (say v) have different colors. Then P' and a segment $P_2(v, v')$ of P_2 from v to v' form an M-alternating path (see Fig. 5(a)), which can be regarded as a second ear other than P_2 in the interior of $P + P_2$, a contradiction.

Case 3: Both u' and v' are internal vertices of P_2. Without loss of generality, assume that u' is nearer to u than v' in P_2. There are two subcases to be considered.

Subcase 3.1: u and u' are of the same color. Then both $P_2(u, u')$ and $P_2(v, v')$ are of even lengths. Then $P_2(u, u') + P' + P_2(v, v')$ is an M-alternating path and thus a second ear in the interior of $P + P_2$ (see Fig. 5(b)), contradicting the minimality of P_2.

Subcase 3.2: u' and u are of different colors. The end edges of $P_2(u', v')$ belong to M. Hence $P_2(u', v') + P'$ is an M-alternating cycle. So $\bar{M} = M \oplus E(P_2(u', v') + P')$ is also a perfect matching of $G - V(G_1)$. Moreover $P_2(u, u') + P' + P_2(v, v')$ is an $\bar{M}$-alternating path and thus a second ear (see Fig. 5(c)), which also leads to a contradiction. The assertion is thus verified.

Repeating the above procedure $r - 1 = q - p$ times, we get our reducible face decomposition. □

By Theorems 4.1 and 4.3, we have

Corollary 4.4. *Let G be a plane bipartite graph other than K_2. Then G is elementary if and only if G has a reducible face decomposition.*

Definition 4.5. Let G be a plane elementary bipartite graph. Let f be a peripheral face of G and P a common path of the peripheries of f and G such that its internal vertices are of degree 2 and its end vertices are of degree $\geqslant 3$ in G. Let $G - P$ denote the resultant subgraph of G by removing the internal vertices and the edges of P. We call f a reducible face of G and P a reducible chain of G if $G - P$ is elementary.

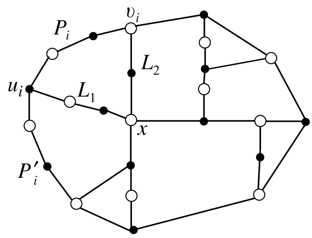


Fig. 6.

It is easily seen that f is a reducible face of G if and only if the intersection of the boundaries of G and f is a path of odd length that is a reducible chain of G.

Theorem 4.6. *Let G be a plane elementary bipartite graph with the cyclomatic number $r = q - p + 1 \geqslant 3$. Then G has at least 2 and at most $q - p$ reducible faces, where q and p denote the numbers of edges and vertices of G, respectively.*

Proof. It suffices to estimate the number of reducible chains. By our RFD Theorem 4.3, $G = G_{r-1} + P_r$, where G_{r-1} is elementary and P_r lies in the exterior of G_{r-1}. Hence P_r is a reducible chain of G. Let C be the boundary of a finite face of G formed by P_r and the periphery of G_{r-1}. Also by Theorem 4.3, there is another RFD $(G_1', \ldots, G_r')$ and $G = G_r' = G_{r-1}' + P_r'$. By the same reason, P_r' is a reducible chain of G other than P_r. Thus G has at least two reducible faces.

Now, we will prove by contradiction that G has at most $q - p$ reducible faces. Suppose that G has $q - p + 1$ reducible faces. Then each finite face of G is reducible. Hence each finite face f of G is a peripheral face and the intersection of the boundaries of G and f is a path of odd length that is a reducible chain of G. That implies that all reducible chains of G form a cycle that is the boundary of G and all vertices lying on the boundary of G are of degree 2 or 3. After removing all reducible chains of G, we get a plane subgraph T of G. It is obvious that T is a plane tree. It follows that the boundary of G passes through all vertices of degree one but no other vertices in T.

A path of a tree T is called a *branch* if its end vertices are of degree 1 and $\geqslant 3$, respectively, and the other vertices are of degree 2. Since the cyclomatic number $r \geqslant 3$, T contains a vertex of degree $\geqslant 3$. We easily see that T has two branches L_1 and L_2 sharing a vertex x with degree $\geqslant 3$ such that the two pendant vertices (of degree 1) u_i and v_i of $L_1 + L_2$ are connected by a reducible chain P_i of G. One of u_i and v_i (say v_i) and x must be of the same color (see Fig. 6). Let $G' = G - P_i'$, where P_i' is a reducible chain of G other than P_i with an end vertex u_i. Let $C = P_i + L_1 + L_2$ be a cycle. All edges between $V(C)$ and $V(G') \backslash V(C) \neq \emptyset$ are incident with only two vertices v_i and x of the same color in C. By Lemma 2.6 such edges are forbidden edges of G'. So G' is not elementary, which contradicts that P_i' is a reducible chain of G. The theorem is proved. □

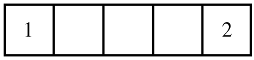


Fig. 7.

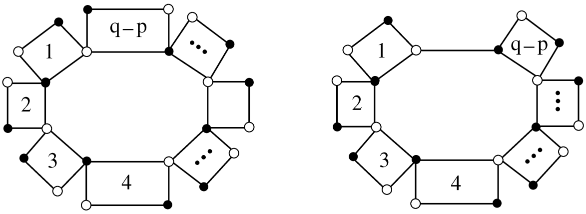


Fig. 8.

Remark 4.7. In Theorem 4.6, the lower and upper bounds for the number of reducible faces are sharp. For example, linear polyomino chains with more than 2 cells, as shown in Fig. 7, have exactly two reducible faces; whereas the graphs shown in Fig. 8 have exactly $q-p$ reducible faces.

Remark 4.8. Obviously, plane elementary bipartite graphs with cyclomatic number $r \in \{1,2\}$ have exactly r reducible faces.

Remark 4.9. A part of Theorem 2.5 generalizes the corresponding results for normal (elementary) hexagonal systems. Cyvin and Gutman [3] once conjectured that normal hexagonal systems have a reducible hexagon, which was proved by He and He [10]. Lately, Zheng [38] obtained a stronger result, i.e. normal hexagonal systems with more than one hexagon have at least two reducible hexagons.

5. Construction for special types of plane elementary bipartite graphs

In the previous section we have actually given a good construction method for plane elementary bipartite graphs, namely starting with a certain face, then adding one new face at each step, we can finally arrive at any plane elementary bipartite graph. We now turn to special types of plane elementary bipartite graphs whose Z-transformation graphs (defined in Section 3) contain a vertex of degree one and, in particular, are paths, respectively. It will be seen that some simple conditions should be satisfied. We need some further terminology.

In what follows, all cycles considered are assumed to be oriented clockwise. This convention will play an important role in our proof of some main theorems in the following. Let M be a perfect matching of G. An edge of G is called an *M-double edge* if it belongs to M and an *M-single edge* otherwise. An M-alternating cycle of G is said to be *proper* [33] if each M-double edge of C goes from a white vertex to a black vertex, and thus each M-single edge of C goes from a black vertex to a white vertex; otherwise C is called an *improper* M-alternating cycle. Similarly, we may define proper and improper M-alternating paths in a cycle. It is obvious that a cycle of G is a proper (improper) M-alternating cycle if and only if every path contained in it is a proper (improper) M-alternating path. Note that a common M-alternating path of two finite faces is proper in the boundary of one of the faces and improper in the boundary of the other one.

Theorem 5.1. *Let G be a plane elementary bipartite graph. Then $Z(G)$ has a vertex of degree one if and only if G has an RFD $(G_1, G_2, \ldots, G_r(=G))$ such that each ear P_i starts with a black vertex and ends with a white vertex or each ear P_i starts with a white vertex and ends with a black vertex with respect to the clockwise orientation of the periphery of G_i, $2 \leqslant i \leqslant r$.*

Proof. *Sufficiency*: If each ear P_i starts with a black vertex and ends with a white vertex ($2 \leqslant i \leqslant r$), it is easy to construct a perfect matching M of G such that G_1 is a proper M-alternating cycle and each ear P_i is an M-alternating path. The finite face of G_1 is an M-resonant face of G. We assert that to M there corresponds a vertex of degree one in $Z(G)$, i.e. G has exactly one M-resonant face. Let R_i denote the boundary (i.e. ring) of the face formed by P_i and a part P'_{i-1} of the periphery of G_{i-1}, $i = 2, \ldots, r$. We can show inductively that the periphery of G_i is a proper M-alternating cycle. Hence P'_i and P_{i+1} are improper and proper M-alternating paths in R_{i+1}, respectively, for $i = 1, \ldots, r-1$. Thus R_{i+1}, $i = 1, \ldots, r-1$, is not an M-alternating cycle. The assertion is verified. If each ear P_i starts with a white vertex and ends with a black vertex ($2 \leqslant i \leqslant r$), it suffices to construct a perfect matching M of G such that G_1 is an improper M-alternating cycle (ring) and each ear P_i is an M-alternating path.

Necessity: Assume that G has a perfect matching M such that G has exactly one M-alternating ring R_1. Assume that R_1 is a proper M-alternating ring. If $r = 1$, the result is trivial. So assume that $r \geqslant 2$. Let $G_1 = R_1$. Among all ear decompositions $G = G_1(=R_1) + P_2 + \cdots + P_r$ such that all the P_i are M-alternating paths, we choose one such that P_2 is a minimal second ear, i.e. no other second ear lies in the interior region surrounded by P_2 and the part P of the periphery of G_1 that lies in the exterior of G_1. We assert that $P + P_2$ is the boundary of a face (ring) of G. Otherwise, by the proof of Theorem 4.3, we know that only Subcase 3.2 could occur. But in this case there exists an M-alternating cycle in the interior of $P + P_2$. By Corollary 3.4, G has two M-resonant faces, a contradiction. On the other hand, we immediately see that P_2 starts with a black vertex and ends with a white vertex in the periphery of $G_2 = G_1 + P_2$. Repeating the above procedure we arrive at an RFD $(G_1, G_2, \ldots, G_r)$ of G such that each ear P_i starts with a black vertex and ends with a white vertex with respect to the orientation of the periphery of G_i, $2 \leqslant i \leqslant r$. If R_1 is an improper M-alternating ring, similarly, we obtain an RFD $(G_1, G_2, \ldots, G_r)$ of G such that each ear P_i starts with a white vertex and ends with a black vertex, $2 \leqslant i \leqslant r$. $\square$

As an immediate consequence, we have

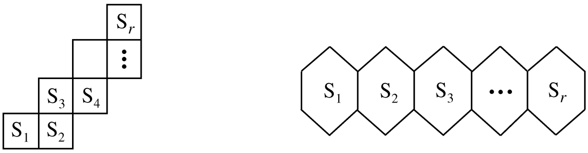


Fig. 9.

Corollary 5.2. *Let G be a plane elementary bipartite graph. Then to M there corresponds a vertex of degree one in* $Z(G)$ *if and only if G has an RFD* $(G_1, G_2, \ldots, G_r)$ *such that the periphery of each* $G_i (1 \leqslant i \leqslant r)$ *is an M-alternating cycle.*

Theorem 5.1 can be used directly to formulate a characterization of hexagonal systems with Z-transformation graph containing a vertex of degree one, which is due to Zhang et al. [27].

For hexagonal systems, the Z-transformation graphs are paths if and only if G is a linear hexagonal chain. For general plane elementary bipartite graphs, the situation appears to be complicated. Now, we give a characterization of plane elementary bipartite graphs whose Z-transformation graphs are paths by using the above result. The proof is tedious and omitted here.

Theorem 5.3. *Let G be a plane elementary bipartite graph. Then* $Z(G)$ *is a path if and only if G has an RFD* $(G_1, G_2, \ldots, G_r)$ *associated with the face sequence* $s_1, s_2, \ldots, s_r$ *and the ear sequence* $P_1, P_2, \ldots, P_r$ *such that*

(1) *the* P_i*'s start with black (resp. white) vertices and end with white (resp. black) vertices with respect to the clockwise orientation of the boundaries of the* G_i*'s;*

(2) s_i *and* s_{i+1} *have edges in common for all i; and*

(3) s_1 *is a peripheral face of* $G_r (=G)$ *or* G_{r-1}.

By Theorem 5.3 and the properties of polyomino and polyhex graphs, we immediately obtain the following result.

Corollary 5.4. *Let G be a polyomino or polyhex graph with perfect matchings. Then* $Z(G)$ *is a path if and only if G is either a single zigzag polyomino chain or a linear polyhex chain as shown in Fig. 9*

6. Forcing edges

Stimulated by some chemical and physical problems, the concept of forcing edges of a hexagonal system was first introduced by Harary et al. [9]. It is natural to ask the following question: what kinds of hexagonal systems have a forcing edge? This problem has been completely solved by Zhang and Li [29] and Hansen and Zheng [8]. We now consider the same problem for general plane bipartite graphs. We first introduce the definition of a forcing edge.

Definition 6.1. Let G be a plane bipartite graph with perfect matchings. An edge of G is called a forcing edge if it is contained in precisely one perfect matching of G.

Lemma 6.2. *Let G be a plane bipartite graph with a perfect matching M. Then $e \in M$ is a forcing edge if and only if each M-alternating cycle passes through the edge e.*

The simple proof is left to the reader.

Theorem 6.3. *Let G be a plane elementary bipartite graph. Then G has a forcing edge if and only if one of the following statements hold*:

(i) *$Z(G)$ has a vertex M of degree one such that the unique M-resonant face is a peripheral face of G,*

(ii) *$Z(G)$ has a vertex M of degree two such that the two M-resonant faces of G have a path in common and the periphery of G is not an M-alternating cycle.*

Proof. *Necessity*: Let G be a plane elementary bipartite graph and e a forcing edge of G belonging to a perfect matching M of G and let $d(M)$ denote the degree of M in $Z(G)$. We claim that $d(M) \leqslant 2$. Otherwise, there exist three distinct M-resonant faces of G. By Lemma 3.7, there are at least two disjoint M-alternating rings. By Lemma 6.2, e is not a forcing edge of G, a contradiction. There are two cases to be considered. If $d(M)=1$, by Corollary 5.2 the periphery of G is an M-alternating cycle. Hence the unique M-resonant face of G is a peripheral face of G. If $d(M)=2$, by Lemma 6.2(ii) follows immediately.

Sufficiency: (1) Assume that $Z(G)$ has a vertex M of degree one such that the unique M-resonant face f of G is a peripheral face of G. Let P denote a component of the intersection of the peripheries of G and f. Since the periphery of G is an M-alternating cycle by Corollary 5.2, P is an M-alternating path the end edges of which belong to M. Let C be any M-alternating cycle. By Corollary 3.4, there exists a unique M-resonant face in the interior of C. Hence C must pass through the path P. By Lemma 6.2, the edges of P belonging to M are forcing edges of G.

(2) Assume that $Z(G)$ has a vertex M of degree two such that the two M-resonant faces intersect and the periphery of G is not an M-alternating cycle. Let P denote a component of the intersection of the two faces resonant with respect to M. Then P is also an M-alternating path the end edges of which belong to M. We claim that each M-alternating cycle C must pass through the path P. Otherwise, suppose that C does not pass through the path P. By Corollary 3.4 it is easily seen that the interior of C contains the two M-resonant faces. Let $I[C]$ denote the subgraph of G formed by C together with its interior. If $G \neq I[C]$, obviously there exists an M-alternating path P' in the exterior of C such that only its two end vertices belong to C. Let C' denote a cycle the interior of which lies in the exterior of C formed by P' and a part of C. Since there is no M-resonant face in the exterior of C, C' is not M-alternating. Hence $C \oplus C'$ is an M-alternating cycle the interior of which properly contains the interior

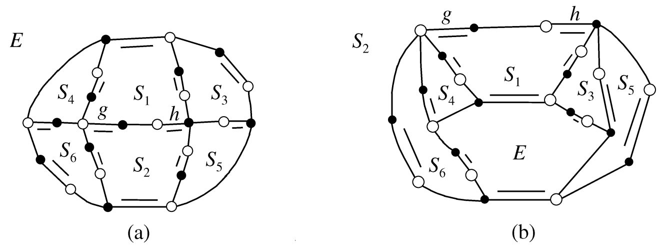


Fig. 10. (a). A plane elementary bipartite graph G with forcing edges g and h; (b). A stereographic projection of G with an RFD with the face sequence $S_1S_3S_4ES_5S_6$.

of C. Repeating the procedure, we finally arrive at the conclusion that the periphery of G is an M-alternating cycle, a contradiction. Our claim is verified. Hence the edges of P belonging to M are forcing edges of G. □

It was proved independently in 1994 and 1995 [29,8] that a hexagonal system has a forcing edge if and only if Part (i) of Theorem 6.3 holds. For a plane elementary bipartite graph, Case (ii) may happen. The perfect matching of a plane elementary bipartite graph shown in Fig. 10, for example, corresponds to a vertex of $Z(G)$ with degree 2 and both the M-double edges g and h are forcing. However, we easily see that properties (i) and (ii) are equivalent if G is considered as embedded in the sphere. Hence Theorem 6.3 can be formulated in a simple form:

Theorem 6.3′. *A plane elementary bipartite graph G has a forcing edge if and only if G has a perfect matching M such that G has exactly two M-resonant faces* (*the exterior face is allowed*) *and their boundaries intersect.*

Furthermore, from Theorem 5.1 we obtain a very simple construction for a plane elementary bipartite graph containing a forcing edge.

Theorem 6.4. *Any plane elementary bipartite graph containing a forcing edge can be obtained by means of the following procedure*:

Step 1: *Construct an ear structure* $G=(x+P_1)+P_2+\cdots+P_r$ *that is an RFD of G satisfying*

(i) *the ears* P_i *start with black* (*resp. white*) *vertices and end with white* (*resp. black*) *vertices with respect to the clockwise orientation of the periphery of* $G_i=(x+P_1)+P_2+\cdots+P_i$, $i=2,\ldots,r$.

(ii) *the face surrounded by* $x+P_1$ *is a peripheral face of G.*

Step 2: *Make a stereographic projection of G.*

This theorem is exemplified in Fig. 10.

We conclude this paper with an important note on the above consequences. In Sections 5 and 6 we considered only plane elementary bipartite graphs. In fact, the following result shows that the same problems for general plane bipartite graphs have also been solved.

Theorem 6.5. *Let G be a plane bipartite graph with perfect matchings and minimum degree* $\geqslant 2$. *If G has a forbidden edge, then G has no forcing edges.*

Proof. Deleting all forbidden edges of G, we obtain a subgraph of G consisting of components $G_1, G_2, \ldots, G_k$ with more than one edge and some K_2's. Since G has a forbidden edge and the degree of each vertex of G is at least two, by the Dulmage–Mendelsohn decomposition of a bipartite graph [18, Chapter 4] we have that $k \geqslant 2$. For any perfect matching M of G, each G_i contains an M-alternating cycle. Hence by Lemma 6.2 G has no forcing edges. □

7. For further reading

The following reference is also of interest to the reader: [23].

Acknowledgements

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References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, American Elsevier, New York, Macmillan, London, 1976.
[2] E. Clar, The Aromatic Sextet, Wiley, London, 1972.
[3] S.J. Cyvin, I. Gutman, Kekulé Structures in Benzenoid Hydrocarbons, Lectures Notes in Chemistry, Vol. 46, Springer, Berlin, 1988.
[4] J.R. Dias, Study of the origin of subspectrality in molecular graphs, Theory Chim. Acta 76 (1989) 153–171.
[5] X. Guo, F. Zhang, k-cycle resonant graphs, Discrete Math. 135 (1994) 113–120.
[6] I. Gutman, S.J. Cyvin, Introduction to the Theory of Benzenoid Hydrocarbons, Springer, Berlin, 1989.
[7] I. Gutman, S.J. Cyvin, Advances in the Theory of Benzenoid Hydrocarbons, Springer, Berlin, 1990.
[8] P. Hansen, M. Zheng, Bonds fixed by fixing bonds, J. Chem. Inform. Comput. Sci. 34 (1994) 297–304.
[9] F. Harary, D.J. Klein, T.P. Zivković, Graphical properties of polyhexes: perfect matching vector and forcing, J. Math. Chem. 6 (1991) 295–306.
[10] W.C. He, W.J. He, Some topological properties of normal benzenoids and coronoids, Match 25 (1990) 225–236.
[11] G. Hetyei, Rectangular configurations which can be covered by 2×1 rectangles, Pécsi Tan. Főisk. Közl 8 (1964) 351–367.
[12] P. John, Eine erste graphentheoretische Bemerkung zur Resonanztheorie benzenoider Kohlenwasserstoffe, Z. Phys. Chem.-Leipzig 270 (1989) 1023–1030.
[13] P. John, Eine zweite graphentheoretische Bemerkung zur Resonanztheorie benzenoider Kohlenwasserstoffe, Match (Comm. Math. Chem.) 26 (1991) 155–170.

[14] P.E. John, Eine dritte graphentheoretische Bemerkung zur Resonanztheorie benzenoider Kohlenwasserstoffe, Match (Comm. Math. Chem.) 28 (1991) 165–180.
[15] P.W. Kasteleyn, The statistics of dimers on a lattice. I., The number of dimer arrangements on a quadratic lattice, Physica 27 (1961) 1209–1225.
[16] D. König, Line systems and determinant, Math. Termész. Ért. 33 (1915) 221–229.
[17] L. Lovász, M.D. Plummer, On minimal elementary bipartite graphs, J. Combin. Theory Ser. B 23 (1977) 127–138.
[18] L. Lovász, M.D. Plummer, Matching Theory, Annals of Discrete Mathematics, Vol. 29, North-Holland, Amsterdam, 1986.
[19] M.D. Plummer, On n-extendable graphs, Discrete Math. 31 (1980) 201–210.
[20] M.D. Plummer, A theorem on matchings in the plane, Graph Theory in Memory of G.A. Dirac, Ann. Discrete Math. 41 (1989) 347–354.
[21] M.D. Plummer, Extending matchings in graphs: a survey, Discrete Math. 127 (1994) 277–292.
[22] M. Randić, On the role of Kekulé valence of structure, Pure Appl. Chem. 55 (2) (1988) 347–354.
[23] H. Sachs, Perfect matchings in hexagonal systems, Combinatorica 4 (1) (1980) 89–99.
[24] N. Trinajstić, Chemical Graph Theory, CRC Press, Boca Raton, FL, 1983.
[25] W.T. Tutte, Graph Theory, Addison-Wesley Publishing Company, Menlo Park, London, 1984.
[26] F. Zhang, R. Chen, When each hexagon of a hexagonal system covers it, Discrete Appl. Math. 30 (1991) 63–75.
[27] F. Zhang, X. Guo, R. Chen, Z-transformation graphs of perfect matchings of hexagonal systems, Discrete Math. 72 (1988) 405–415.
[28] F. Zhang, X. Guo, R. Chen, The connectivity of Z-transformation graphs of perfect matchings of hexagonal systems, Acta Math. Appl. Sinica 4 (2) (1988) 131–135.
[29] F. Zhang, X. Li, Hexagonal systems with forcing edges, Discrete Math. 140 (1995) 253–263.
[30] F. Zhang, Y. Liu, Estimation of the resonance energy of benzenoid hydrocarbons, Chinese Sci. Bull. 38 (1993) 1391–1394 (in Chinese).
[31] F. Zhang, M. Zheng, Generalized hexagonal systems with each hexagon being resonant, Discrete Appl. Math. 36 (1992) 67–73.
[32] H. Zhang, The connectivity of Z-transformation graphs of perfect matchings of polyominoes, Discrete Math. 158 (1996) 257–272.
[33] H. Zhang, F. Zhang, The rotation graphs of perfect matchings of plane bipartite graphs, Discrete Appl. Math. 73 (1997) 5–12.
[34] H. Zhang, F. Zhang, Perfect matchings of polyomino graphs, Graphs Combin. 13 (1997) 295–304.
[35] H. Zhang, F. Zhang, Block graph of Z-transformation graph of perfect matchings of plane elementary bipartite graphs, Ars Combin. 53 (1999) 309–314.
[36] M. Zheng, The k-resonant benzenoid systems, J. Mol. Struct. (Theochem) 231 (1991) 321–334.
[37] M. Zheng, Construction of 3-resonant benzenoid systems, J. Mol. Struct. (Theochem) 277 (1992) 1–14.
[38] M. Zheng, Perfect matchings in hexagonal systems, Doctoral Thesis, Rutgers University, 1992.